# Solution of functional equations and reduction of dimension in the local energy transfer theory of incompressible, three-dimensional turbulence

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It is shown that the set of integrodifferential and algebraic functional equations of the local energy transfer theory may be considerably reduced in dimension for the case of isotropic turbulence. This is achieved without restricting the solution space. The basis for this is a complete analytical solution to the functional equations $Q(k;t,t')=H(k;t,t')Q(k;t',t')$ and $H(k;t,s)H(k;s,t')=H(k;t,t')$. The solution is proved to depend only on a single function $\phi(k;t)$ solely determining $Q$ and $H$. Hence the dimension of both the dependent and the independent variables is reduced by one. From the latter, the corresponding two integrodifferential equations are lowered to a single integrodifferential equation for $\phi(k;t)$, extended by an integral side condition on the $k$ dependence of $\phi(k;t)$. In the limit $\nu\to 0$, a partial solution to the reduced set of equations is presented in the Appendix.

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## I. INTRODUCTION

In the context of fluid turbulence, a certain class of theories has been developed that is formulated in wave-number space and relies on the truncated renormalized series expansion of the nonlinear convection term. The idea originated in the pioneering work of Kraichnan [1], Edwards [2] and Herring [3,4], although these early theories were incompatible with the Kolmogorov power law [5,6]. There have been numerous later theories that use the Lagrangian coordinate system but we are concerned here with the local energy transfer (LET) theory that is unique in yielding Kolmogorov behavior in an Eulerian framework. An overview of the topic is given in [6].

All these theories have in common their transport equations that constitute a nonlinear integrodifferential equation depending on the wave number $k=|\mathbf{k}|$, time $t$, and an additional delay time $t'$. It is particularly due to the latter's dependence on $t'$ that numerical computations may become forbiddingly expensive. In fact, this is the primary reason, that such theories have almost exclusively been applied to homogeneous isotropic turbulence. The key difficulty with respect to the $t'$ dependence is that at each time step a field depending on $k$ and $t'$ has to be stored, where $t'$ varies between 0 and $t$. Hence, as time proceeds, increasingly larger two-dimensional fields have to be kept in memory, from the beginning up to the current time $t$. For this reason, usually only very few ''eddy turnover'' times may be computed.

In the following sections it is shown that the structure of the LET equations is such that in the case of isotropic turbulence, the dimensionality of both the dependent and the independent variables may be reduced by one. This goal is achieved without loss of information. Hence a *single* field has to be stored depending solely on $k$ and $t$.

## II. REDUCTION OF THE LET EQUATIONS

The LET equations in their most up to date form may be found in [7].

### A. Solution of the $Q$ and $H$ functional equations

In the LET theory, a tensor-valued quantity $\mathbf{H}$, called the propagator, is defined; this relates the correlation tensor

$$Q_{ij}(\boldsymbol{k};t,t')=\langle u_i(\boldsymbol{k},t)u_j(-\boldsymbol{k},t')\rangle, \tag{1}$$

to itself at different instants of time ($t$, $t'$ where $t>t'$), thus

$$Q_{ij}(\boldsymbol{k};t,t')=H_{im}(\boldsymbol{k};t,t')Q_{mj}(\boldsymbol{k};t',t'). \tag{2}$$

A functional equation for $\mathbf{H}$ is given by

$$H_{im}(\boldsymbol{k};t,s)H_{mj}(\boldsymbol{k};s,t')=H_{ij}(\boldsymbol{k};t,t'), \tag{3}$$

$$H_{ij}(\boldsymbol{k};t,t)=D_{ij}(\boldsymbol{k}), \tag{4}$$

where $D_{ij}(\boldsymbol{k})$ is the projection operator

$$D_{ij}(\boldsymbol{k})=\delta_{ij}-\frac{k_ik_j}{|\boldsymbol{k}|^2}. \tag{5}$$

The equations (2) and (3) imply a certain restriction on $\mathbf{Q}$ and $\mathbf{H}$ that will be explored below for the case of isotropic turbulence. Under this assumption, $\mathbf{H}$ may be written as

$$H_{ij}(\boldsymbol{k};t,t')=D_{ij}(\boldsymbol{k})H(k;t,t'). \tag{6}$$

Substituting Eqs. (5) and (6) in Eq. (3), we find the scalar functional equation

$$H(k;t,s)H(k;s,t')=H(k;t,t'), \tag{7}$$

where $k=|\boldsymbol{k}|$. A similar scalar equation may be derived from Eq. (2) by invoking the isotropy condition

$$Q_{ij}(\boldsymbol{k};t,t')=D_{ij}(\boldsymbol{k})Q(k;t,t'). \tag{8}$$

Using Eqs. (8), (2) and (6) we derive the scalar functional equation

 

$$Q(k;t,t')=H(k;t,t')Q(k;t',t'), \tag{9}$$

which is often called the fluctuation-dissipation relation. From Eqs. (4) and (6), it follows that

$$H(k;t,t)=1. \tag{10}$$

In the following, we derive a general solution to the Eqs. (7) and (9), the result also being consistent with Eq. (10). For this purpose we define a new function $\mathcal{H}$ given by

$$\mathcal{H}(k;t,t')=\ln[H(k;t,t')]. \tag{11}$$

Taking the logarithm of Eq. (7) and implementing Eq. (11) we find

$$\mathcal{H}(k;t,s)+\mathcal{H}(k;s,t')=\mathcal{H}(k;t,t'). \tag{12}$$

It should be noted that in principle $H(k;t,t')$ is a positive function. However, in some numerical calculations negative values have been found (see, e.g., [7]). In order to avoid any restriction on $H(k;t,t')$, we allow for complex values of $\mathcal{H}(k;t,t')$.

In a second step, we differentiate Eq. (12) with respect to $t$ to obtain

$$\frac{\partial\mathcal{H}(k;t,s)}{\partial t}=\frac{\partial\mathcal{H}(k;t,t')}{\partial t}. \tag{13}$$

It should be noted that any differential consequence of Eq. (12) may not stand, since integration with respect to $t$ may not necessarily lead back to Eq. (12). Hence, any consequences of Eq. (13) have to be validated against Eq. (12).

Apparently both sides of Eq. (13) depend on $k$ and $t$ while the left-hand side also depends on $s$ and the right-hand side has an additional $t'$ dependence. Thus, we have to ensure that both sides can only depend on $k$ and $t$ to make Eq. (13) true. As an immediate result we conclude that

$$\frac{\partial\mathcal{H}(k;t,s)}{\partial t}=\frac{\partial\mathcal{H}(k;t,t')}{\partial t}\equiv f(k;t). \tag{14}$$

Considering only the equivalence to the right, we may integrate with respect to $t$ to find

$$\mathcal{H}(k;t,t')=h_1(k;t)+h_2(k;t'), \tag{15}$$

where $h_1=\int f dt$ and $h_2$ are arbitrary functions of the arguments. As mentioned above, the result has to be cross-checked with the original equation (12). Substituting Eq. (15) in Eq. (12), we obtain

$$h_1(k;t)+h_2(k;s)+h_1(k;s)+h_2(k;t')=h_1(k;t)+h_2(k;t'). \tag{16}$$

Except for the second and third terms on the left-hand side, all expressions cancel. Hence we obtain the relation between $h_1$ and $h_2$,

$$h_2(k;s)=-h_1(k;s)\equiv -h(k;s). \tag{17}$$

Substituting Eqs. (17) and (15) in Eq. (11), we obtain the general solution of Eq. (7),

$$H(k;t,t')=\mathrm{e}^{h(k;t)-h(k;t')}\equiv\frac{\phi(k;t)}{\phi(k;t')}. \tag{18}$$

Condition (10) is also solved identically. The function $\phi(k;t)$ has been introduced to show that $H(k;t,t')$ is determined as a ratio of one function to itself at different times.

Solution (18) also illustrates the symmetry transformation (see, e.g., [8]) of Eq. (7)

$$\tilde{H}(k;t,t')=H(k;t,t')\mathrm{e}^{h(k;t)-h(k;t')}, \tag{19}$$

such that Eq. (7) is form invariant under the transformation $H\rightarrow\tilde{H}$. The latter symmetry also holds true for the nonisotropic functional equation (3).

For the purpose of finding a general solution to Eq. (9), we define a new function

$$\mathcal{Q}(k;t,t')=\ln[Q(k;t,t')]. \tag{20}$$

Although we will see later that $Q(k;t,t')$ is in principle always positive, the following analysis holds true for negative $Q(k;t,t')$ if $\mathcal{Q}(k;t,t')$ is allowed to admit complex values. Strictly speaking *a priori* it is only known that the matrix $Q_{ij}(k;t,t')$ is positive semidefinite in the continuum indices $t$ and $t'$.

Substituting Eqs. (18) and (20) in Eq. (9) and taking the logarithm, we find

$$\mathcal{Q}(k;t,t')=h(k;t)-h(k;t')+\mathcal{Q}(k;t',t'). \tag{21}$$

As done previously, we take the derivative with respect to $t$ to obtain

$$\frac{\partial\mathcal{Q}(k;t,t')}{\partial t}=\frac{\partial h(k;t)}{\partial t}\equiv g(k;t). \tag{22}$$

As an immediate result we obtain

$$\mathcal{Q}(k;t,t')=h(k;t)+q(k;t'), \tag{23}$$

where $q(k;t')$ appears due to the integration. Substituting the latter back in Eq. (21) we find it is identically solved. Employing the definition of $\mathcal{Q}(k;t,t')$ we obtain the solution

$$Q(k;t,t')=e^{h(k;t)+q(k;t')}\equiv\phi(k;t)\psi(k;t'). \tag{24}$$

Here $\psi$ has been introduced to illuminate the product structure of $Q$.

The final form of $Q(k;t,t')$ may be obtained by invoking the symmetry in $t$ and $t'$, i.e.,

$$Q(k;t,t')=Q(k;t',t). \tag{25}$$

Substituting Eq. (24) in Eq. (25), we obtain after rearranging terms

$$\frac{\phi(k;t)}{\psi(k;t)}=\frac{\phi(k;t')}{\psi(k;t')}\equiv\frac{1}{\gamma(k)}, \tag{26}$$

where $\gamma(k)$ has been introduced by the same arguments as above. We finally obtain

$$\psi(k;t)=\phi(k;t)\gamma(k). \tag{27}$$

In fact $\gamma(k)$ can be set equal to 1 without loss of generality, the reason being the following: Since $Q(k;t,t')=\phi(k;t)\phi(k;t')\gamma(k)$, we may absorb $\sqrt{\gamma(k)}$ in $\phi(k;t)$ calling it $\tilde{\phi}(k;t)=\phi(k;t)\sqrt{\gamma(k)}$ for the moment. The factor $\gamma(k)$ can only be positive because in the limit $t\to t'$ the quantities $Q(k;t,t)$ and $\phi(k;t)^2$ are always positive. Introducing $\tilde{\phi}(k;t)$ in Eq. (18), we find that $\sqrt{\gamma(k)}$ cancels. Most importantly, the same is also true for the evolution equations for $\phi(k;t)$ to be derived subsequently. Hence, we may set $\gamma(k)=1$ and the final solution of the Eqs. (7), (9), and (25) is given in terms of the new function $\phi(k;t)$

$$Q(k;t,t')=\phi(k;t)\phi(k;t'), \tag{28}$$

$$H(k;t,t')=\frac{\phi(k;t)}{\phi(k;t')}. \tag{29}$$

Again, also the last part of this analysis accounts for negative values of the dependent and independent variables (as found in some numerical simulations) if complex variables are employed.

### B. Derivation of the reduced LET integro-differential equations

In order to show consistency of the above solutions with the LET transport equations, we may first give their unrestricted form for isotropic turbulence. The two-point two-time correlation equation given in [7]

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)Q(k;t,t') = \int L(\boldsymbol{k},\boldsymbol{j})\left[\int_0^{t'} H(k;t',t'')Q(j;t,t'')Q(|\boldsymbol{k}-\boldsymbol{j}|;t,t'')dt'' - \int_0^{t} H(j;t,t'')Q(k;t',t'')Q(|\boldsymbol{k}-\boldsymbol{j}|;t,t'')dt''\right]d^3j, \tag{30}$$

along with the energy equation

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)Q(k;t,t)=2\int L(\boldsymbol{k},\boldsymbol{j})\int_0^t Q(|\boldsymbol{k}-\boldsymbol{j}|;t,t'')\times[H(k;t,t'')Q(j;t,t'') - H(j;t,t'')Q(k;t,t'')]dt''d^3j, \tag{31}$$

and Eqs. (7), (9), and (25) form a closed set of equations where

$$L(\boldsymbol{k},\boldsymbol{j})=\frac{[\mu(k^2+j^2)-kj(1+2\mu^2)](1-\mu^2)kj}{k^2+j^2-2kj\mu},$$

$$\mu=\cos\Theta_{\boldsymbol{k}\angle\boldsymbol{j}}=\frac{\boldsymbol{k}\cdot\boldsymbol{j}}{|\boldsymbol{k}||\boldsymbol{j}|}. \tag{32}$$

Note that Eq. (31) has a slightly different structure than Eq. (30) since factors of 2 appear on both sides. These factors are due to the $t$-derivatives in Eq. (30) that change their structure if Eq. (31) is derived in the limit $t'\to t$.

Since a dimensional reduction for $Q(k;t,t')$ and $H(k;t,t')$ has been accomplished due to Eqs. (28) and (29) we may in the final step derive the LET equation for $\phi(k;t)$. Equations (30) and (31), respectively, reduce to

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\phi(k;t) = \int L(\boldsymbol{k},\boldsymbol{j})\left[\int_0^{t'}\frac{\phi(j;t)\phi(j;t'')\phi(|\boldsymbol{k}-\boldsymbol{j}|;t)\phi(|\boldsymbol{k}-\boldsymbol{j}|;t'')}{\phi(k;t'')}\times dt'' - \int_0^{t}\frac{\phi(j;t)\phi(k;t'')\phi(|\boldsymbol{k}-\boldsymbol{j}|;t)\phi(|\boldsymbol{k}-\boldsymbol{j}|;t'')}{\phi(j;t'')}\times dt''\right]d^3j, \tag{33}$$

and

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\phi(k;t) = \int L(\boldsymbol{k},\boldsymbol{j})\int_0^t \phi(|\boldsymbol{k}-\boldsymbol{j}|;t)\phi(|\boldsymbol{k}-\boldsymbol{j}|;t'')\phi(j;t)\times\left[\frac{\phi(j;t'')}{\phi(k;t'')}-\frac{\phi(k;t'')}{\phi(j;t'')}\right]dt''d^3j. \tag{34}$$

Note that the factors of 2 in Eq. (31) have canceled out and no longer appear in Eq. (34).

Equation (34) is the only equation left for the evolution of the quantity $\phi(k;t)$ and is fully consistent with Eq. (33). It is an easy matter to show that Eq. (33) reduces to Eq. (34) in the limit $t'\to t$.

Though consistency between Eq. (33) and (34) is apparent we may still derive an additional condition from Eq. (33). Note that the left-hand side solely depends on $k$ and $t$ while the right-hand side possesses an additional $t'$ dependence. Taking the derivative of Eq. (33) with respect to $t'$ and multiplying by $\phi(k;t')$, we obtain the integral equation

$$\int L(\boldsymbol{k},\boldsymbol{j})\phi(j;t)\phi(j;t')\phi(|\boldsymbol{k}-\boldsymbol{j}|;t)\phi(|\boldsymbol{k}-\boldsymbol{j}|;t')d^3j=0, \tag{35}$$

which holds for arbitrary $t$ and $t'$ and gives an additional constraint on the structure of $\phi(k;t)$. We conclude that in the case of isotropic turbulence the LET equations have been reduced to Eqs. (34) and (35).

It is still very difficult to obtain analytical solutions to the nonlinear integro-differential equation (34) with the additional constraint (35). Up to now only a partial solution has been obtained in the limit $\nu\to 0$. However, this partial solution has no physical significance since it possesses a finite time singularity. For this reason, this result has been put in the Appendix.

## III. SUMMARY AND CONCLUSIONS

A method is presented to derive the complete solution of the algebraic functional equations for $Q$ and $H$ in the LET theory for isotropic turbulence in terms of a single lower-dimensional function. A reduction in the number of independent variables has been achieved. More importantly, a reduction in dimension by one has been accomplished without limiting the space of solutions.

In the second step the corresponding integrodifferential equations of the LET theory have been reduced to a single equation plus an integral side condition. From a practical point of view, the present results for the isotropic functions $Q$ and $H$ may serve to reduce computational costs considerably since keeping $Q(k;t,t')$ and $H(k;t,t')$ in memory has been boiled down to only storing $\phi(k;t)$. This raises the possibility that one might test the theory on inhomogeneous and shear flows.

## APPENDIX: PARTIAL SOLUTION IN THE LIMIT $\nu\to 0$

Due to the integrals in Eqs. (34) and (35), it is difficult to apply group theoretical methods to obtain analytical solutions. For this reason we have found only one partial solution.

Suppose $\phi(k;t)$ has a product structure of the form

$$\phi(k;t)=f(k)g(t). \qquad \text{(A1)}$$

Imposing the zero viscosity limit $\nu\to 0$ we obtain, respectively, from Eqs. (34) and (35)

$$f(k)\frac{dg(t)}{dt}=\int_0^t g(t'')dt''g(t)^2\int L(\boldsymbol{k},\boldsymbol{j})f(|\boldsymbol{k}-\boldsymbol{j}|)^2 f(j)\times\left[\frac{f(j)}{f(k)}-\frac{f(k)}{f(j)}\right]d^3j, \qquad \text{(A2)}$$

and

$$g(t)^2g(t')^2\int L(\boldsymbol{k},\boldsymbol{j})f(j)^2f(|\boldsymbol{k}-\boldsymbol{j}|)^2d^3j=0. \qquad \text{(A3)}$$

Supposing $g(t)\neq 0$ for all times we may divide Eq. (A3) by $g(t)$. In turn this may be used to cancel out the first term in brackets on the right-hand side of Eq. (A2). Dividing the resulting equation by $f(k)$ and $\int_0^t g(t'')dt''g(t)^2$, we obtain the separated equation

$$\frac{dg(t)/dt}{\int_0^t g(t'')dt''g(t)^2}=-\int L(\boldsymbol{k},\boldsymbol{j})f(|\boldsymbol{k}-\boldsymbol{j}|)^2d^3j\equiv c_1, \qquad \text{(A4)}$$

where the equality can only be true if both sides equal a constant, here denoted by $c_1$.

Equating the $g$ part on the left-hand side with $c_1$, multiplying through by $\int_0^t g(t'')dt''$, and differentiating with respect to $t$, we obtain the nonlinear second-order ordinary differential equation (ODE)

$$\frac{d^2g(t)}{dt^2}g(t)-2\left(\frac{dg(t)}{dt}\right)^2=c_1g(t)^4. \qquad \text{(A5)}$$

The latter equation admits two symmetries one of which is a scaling symmetry and the other a translation in time. From basic group theory, it is known that a second order ODE, such as Eq. (A5) is solvable in terms of quadratures if it admits at least two symmetry groups [9]. For brevity we introduce a simpler route, which nevertheless, implicitly relies on these two groups. Substituting

$$\frac{dg(t)}{dt}=h(g)\quad\Rightarrow\quad\frac{d^2g(t)}{dt^2}=\frac{dh(g)}{dg}h(g), \qquad \text{(A6)}$$

in Eq. (A5) and integrating the resulting Bernoulli-type first order ODE, we obtain

$$h(g)=\pm g^2\sqrt{2c_1\ln(g)+c_2}, \qquad \text{(A7)}$$

where $c_2$ is a constant of integration.

Substituting the latter result in Eq. (A6) and integrating, we obtain the solution to Eq. (A5) in implicit form

$$\sqrt{\frac{\pi}{2c_1}}e^{c_2/2c_1}\mathrm{erf}\sqrt{\ln[g(t)]+\frac{c_2}{2c_1}}=\pm(t+c_3), \qquad \text{(A8)}$$

where $c_3$ is an additional constant of integration and erf is the error function [10]. Introducing $\mathrm{erf}^{-1}$ as the inverse of erf we acquire the final solution to Eq. (A5)

$$g(t)=\exp\left\{\left[\mathrm{erf}^{-1}\left(\pm\sqrt{\frac{2c_1}{\pi}}e^{-c_2/2c_1}(t+c_3)\right)\right]^2-\frac{c_2}{2c_1}\right\}. \qquad \text{(A9)}$$

Hence the $g$ equation in Eq. (A4) is solved completely. The $f$-equation, a nonlinear integral equation, is considerably more difficult and no analytical solution has been found yet.

In contrast to the Euler equations (the Navier-Stokes equations in the limit $\nu=0$), which admit two scaling groups [11], Eqs. (34) and (35) admit only one scaling group. In particular, Eqs. (34) and (35) do not admit scaling in time. However, one has to draw a distinction between two different cases. Firstly, the Euler equation, which has $\nu=0$ and a dissipation rate $\epsilon=0$. Secondly, the Navier-Stokes equation at infinite Reynolds number, where $\nu\to 0$ in such a manner that the dissipation $\epsilon$ is constant.

The procedure that leads to LET renormalizes a viscous interaction and that renormalization (the time-history integral terms) has to be able to represent the finite-energy-transfer rate which is equal to $\epsilon$ for stationary turbulence. Accordingly, as shown by Edwards [12,13] equation (34) must, in place of the viscous term, contain a delta function of magnitude $\epsilon$ in the limit of infinite Reynolds number.

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